Marginal and technical rates of substitution

Marginal rate of substitution (MRS)

Given a utility function that depends on the quantities of two goods q_1 and q_2, we call the *marginal rate of substitution (MRS)* the rate at which one good can be traded for the other. That is, it represents the amount of one good a consumer is willing to give up to obtain one additional unit of the other good while keeping utility constant.

We compute the differential of the utility function $U(q_1, q_2)$:

$$du = \frac{\partial u}{\partial q_1} dq_1 + \frac{\partial u}{\partial q_2} dq_2$$

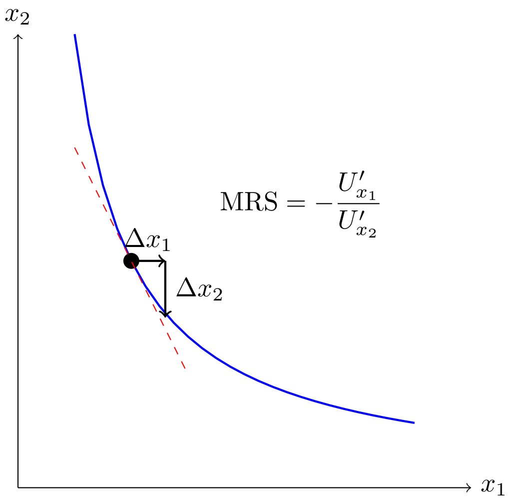
Since $du = 0$ along an indifference curve:

$$\frac{\partial u}{\partial q_1} dq_1 + \frac{\partial u}{\partial q_2} dq_2 = 0 \Rightarrow \frac{dq_2}{dq_1} = -\frac{\frac{\partial u}{\partial q_1}}{\frac{\partial u}{\partial q_2}}$$

The magnitude of this slope represents the marginal rate of substitution between the goods:

$$\text{MRS} = \left| \frac{dq_2}{dq_1} \right| = \frac{\frac{\partial u}{\partial q_1}}{\frac{\partial u}{\partial q_2}}$$

Graphically



Example

Given the utility function $U(q_1, q_2) = 5q_1q_2$, calculate the MRS at the point $(q_1 = 5, q_2 = 2)$.

$$\frac{\partial u}{\partial q_1} = 5q_2 = 5 \cdot 2 = 10$$
$$\frac{\partial u}{\partial q_2} = 5q_1 = 5 \cdot 5 = 25$$
$$\text{MRS} = \frac{10}{25} = 0.4$$

Interpretation: To increase q_1 by one unit, 0.4 units of q_2 must be given up in order to maintain the same level of utility.

If we invert the rate and compute $\frac{dq_1}{dq_2}$, we obtain:

$$\frac{dq_1}{dq_2} = \frac{\frac{\partial u}{\partial q_2}}{\frac{\partial u}{\partial q_1}} = \frac{25}{10} = 2.5$$

Interpretation: To increase q_2 by one unit, 2.5 units of q_1 must be given up to keep utility constant.

Technical rate of substitution (TRS)

Given a production function that depends on two inputs x_1 and x_2, we define the *technical rate of substitution (TRS)* as the rate at which a firm can substitute one input for another while keeping the level of output constant.

We compute the differential of the production function $q(x_1, x_2)$:

$$dq = \frac{\partial q}{\partial x_1} dx_1 + \frac{\partial q}{\partial x_2} dx_2$$

Since $dq = 0$ along an isoquant:

$$\frac{\partial q}{\partial x_1} dx_1 + \frac{\partial q}{\partial x_2} dx_2 = 0 \Rightarrow \frac{dx_2}{dx_1} = -\frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}}$$

The magnitude of this slope represents the technical rate of substitution:

$$\text{TRS} = \left| \frac{dx_2}{dx_1} \right| = \frac{\frac{\partial q}{\partial x_1}}{\frac{\partial q}{\partial x_2}}$$

Example

Given the production function:

$$q(a, b) = 20 - 7a + 8b - a^2 + b^2$$

calculate the TRS at the point $a = 1.2, b = 2.2$.

$$\frac{\partial q}{\partial a} = -7 - 2a = -7 - 2(1.2) = -9.4$$
$$\frac{\partial q}{\partial b} = 8 + 2b = 8 + 2(2.2) = 12.4$$
$$\text{TRS} = \frac{-9.4}{12.4} \approx -0.758$$

Interpretation: To increase a by one unit, approximately 0.758 units of b must be given up to maintain the same level of output.

If we invert the rate and calculate $\frac{dx_1}{dx_2}$, we obtain:

$$\frac{dx_1}{dx_2} = \frac{\frac{\partial q}{\partial b}}{\frac{\partial q}{\partial a}} = \frac{12.4}{-9.4} \approx -1.319$$

Interpretation: To increase b by one unit, approximately 1.319 units of a must be given up to maintain constant output.

Notation

In many texts, a shorthand notation such as $\text{MRS}(x_1/x_2)$ is used to refer to marginal rates of substitution.

$\text{MRS}(x_1/x_2)$ represents the marginal rate of substitution where the numerator is the derivative with respect to x_2, and the denominator is the derivative with respect to x_1, that is:

$$\text{MRS}(x_1/x_2) = \frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$

Interpretation: it indicates how many units of x_2 must be given up to obtain one additional unit of x_1, keeping utility constant.

Conversely, $\text{MRS}(x_2/x_1)$ represents the marginal rate of substitution where the numerator is the derivative with respect to x_1, and the denominator is the derivative with respect to x_2, that is:

$$\text{MRS}(x_2/x_1) = \frac{dx_1}{dx_2} = \frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}}$$

Interpretation: it indicates how many units of x_1 must be given up to obtain one additional unit of x_2, keeping utility constant.

Intuitive interpretation

When an expression like

$$\frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$

appears, it should be interpreted as: "how much x_2 must decrease in order to increase x_1 by one unit, keeping utility constant."

An intuitive way to think about it is:

- The numerator is what is given up
- The denominator is what is being obtained